Fundamental Theorems in W^*-Algebras and the Kaplansky density theorem, II

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Abstract

We shall give extraordinarily elementary proofs of that any W^*-algebra has an identity and the Kaplansky density theorem. We shall reconstruct Section 2 in [1].

1. Second duals of C^*-algebras

We can prove within the limits of the theory of C^*-algebras and without using of representations on Hilbert spaces, that the second dual of a C^*-algebra is a C^*-algebra. By this fact, we immediately see the Kaplansky density theorem. Also, we give another proof of the Kaplansky density theorem in virtue of polar decomposition.

Lemma 1. *The second dual A^{**} of a Banach *-algebra A is a Banach algebra and A is a subalgebra of A^{**}. The involution of A can be extended to the continuous linear mapping $x \mapsto x^*$ on A^{**} by continuity with respect to the $\sigma(A^{**}, A^*)$-topology. Furthermore, it holds that*

(i) *for any $x \in A^{**}$, the linear mapping $A^{**} \ni y \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous;*
(ii) *for any $y \in A$, the linear mapping $A^{**} \ni x \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous;*
(iii) *for any $x \in A^{**}$ and $y \in A$, $x^{**} = x$ and $(xy)^* = y^*x^*$.*

Proof. For an element $x \in A^{**}$, define $x^* \in A^{**}$ by $\langle x^*, \varphi\rangle = \overline{\langle x, \varphi^*\rangle}$ with $\varphi \in A^*$. Obviously, the linear mapping $A^{**} \ni x \mapsto x^* \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous and $x^{**} = x$.

For any $y \in A$ and $\varphi \in A^*$, we can regard $y\varphi \in A^*$ as a $\sigma(A^{**}, A^*)$-continuous linear form on A^{**}. For an element $x \in A^{**}$, since the linear form $A \ni y \mapsto \langle x, y\varphi\rangle$ is bounded, there exists an element $\varphi x \in A^*$ such that $\langle x, y\varphi\rangle = \langle y, \varphi x\rangle$ for every $y \in A$. We have $|\langle y, \varphi x\rangle| \le \|y\varphi\|\|x\| \le \|\varphi\|\|x\|\|y\|$ for every $x \in A^{**}, y \in A$ and $\varphi \in A^*$ and so $\|\varphi x\| \le \|\varphi\|\|x\|$. Regarding $\varphi x \in A^*$ as a $\sigma(A^{**}, A^*)$-continuous linear form on A^{**}, for each x and y in A^{**}, we define a linear form xy on A^* by $\langle xy, \varphi\rangle = \langle y, \varphi x\rangle$ with $\varphi \in A^*$. Since $|\langle xy, \varphi\rangle| \le \|\varphi x\|\|y\| \le \|\varphi\|\|x\|\|y\|$ for every $x, y \in A^{**}$ and $\varphi \in A^*$, xy is bounded on A^* and so $xy \in A^{**}$. By the definitions, for any $x \in A^{**}$, the mapping $A^{**} \ni y \mapsto xy \in A^{**}$ is

$\sigma(A^{**}, A^*)$-continuous and, for any $y \in A$, the mapping $A^{**} \ni x \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous. For any $a, b \in A, x \in A^{**}$ and $\varphi \in A^*$, it holds that

$$(xa)b = (\lim_{u \to x, u \in A} (ua))b = \lim_{u \to x, u \in A} (ua)b = \lim_{u \to x, u \in A} u(ab) = x(ab).$$

Similarly, we have $(xy)z = x(yz)$ for every $x, y, z \in A^{**}$. Moreover, it holds the distributive law. Since the conditions to the norm is clearly satisfied, A^{**} is a Banach algebra and A is a subalgebra of A^{**}. For any $x \in A^{**}$ and $y \in A$, it follows that

$$(xy)^* = \lim_{u \to x, u \in A} (uy)^* = \lim_{u \to x, u \in A} y^* u^* = y^* x^*.$$

□

Proposition 2. *The second dual A^{**} of a C^*-algebra is a Banach *-algebra with an identity. Furthermore, the multiplication in A^{**} is separately continuous with respect to the $\sigma(A^{**}, A^*)$-topology and the involution is $\sigma(A^{**}, A^*)$-continuous.*

Proof. Let $\mathscr{H}_\varphi$ be the Hilbert space associated with a state $\varphi \in S(A)$ and η_φ the canonical mapping of A into $\mathscr{H}_\varphi$. Since $\|\eta_\varphi(x)\| \leq \|x\|$ for every $x \in A$, we can take the transposed mapping ${}^t\eta_\varphi \colon \mathscr{H}_\varphi^* \to A^*$ of η_φ. Since $\|{}^t\eta_\varphi\| \leq 1$, let ${}^{tt}\eta_\varphi$ be the bitranspose $A^{**} \to \mathscr{H}_\varphi^{**} = \mathscr{H}_\varphi$ of η_φ. Since, for any $x \in A^{**}$ and $\xi \in \mathscr{H}_\varphi^*$, $\langle x, {}^t\eta_\varphi(\xi)\rangle = \langle {}^{tt}\eta_\varphi(x), \xi\rangle$, ${}^t\eta_\varphi$ is continuous with respect to the $\sigma(\mathscr{H}_\varphi^*, \mathscr{H}_\varphi)$-topology and $\sigma(A^*, A^{**})$-topology.

Since the unit ball A_1 of A is $\sigma(A^{**}, A^*)$-dense in the unit ball of A^{**}, for any $y \in A$ and any element x of the unit ball of A^{**}, we have

$$|\langle y, \varphi x\rangle| = |\langle xy, \varphi\rangle| = \lim_{u \to x, u \in A_1} |\langle uy, \varphi\rangle| \leq \varphi(uu^*)^{1/2}\varphi(y^*y)^{1/2} \leq \|\eta_\varphi(y)\|.$$

Therefore there exists an element $\xi \in \mathscr{H}_\varphi^*$ such that $\varphi x = {}^t\eta_\varphi(\xi)$ and $\|\xi\| \leq 1$. Hence $\{\varphi x \mid x \in A^{**}, \|x\| \leq 1\}$ is included in the image of the unit ball of $\mathscr{H}_\varphi^*$ under ${}^t\eta_\varphi$. Since the unit ball of $\mathscr{H}_\varphi^*$ is $\sigma(\mathscr{H}_\varphi^*, \mathscr{H}_\varphi)$-compact, the balanced convex set $\{\varphi x \mid x \in A^{**}, \|x\| \leq 1\}$ is relatively compact with respect to the $\sigma(A^*, A^{**})$-topology. For an element $y \in A^{**}$, the linear form $y\varphi\colon A^{**} \ni x \mapsto \langle xy, \varphi\rangle$ belongs to $(A^{**})^*$. Let $\mathfrak{F}$ be a filter on A converging to $y \in A^{**}$ with respect to the $\tau(A^{**}, A^*)$-topology; then the image of $\mathfrak{F}$ under the mapping $u \mapsto u\varphi$ converges uniformly to $y\varphi$ on the unit ball of A^{**}. Since, for any $u \in A$, $u\varphi \in A^*$ is $\sigma(A^{**}, A^*)$-continuous, $y\varphi$ is $\sigma(A^{**}, A^*)$-continuous on the unit ball of A^{**}. Hence, by the Banach theorem, $y\varphi$ is $\sigma(A^{**}, A^*)$-continuous on A^{**}, that is, $y\varphi \in A^*$. Therefore the mapping $A^{**} \ni x \mapsto \langle xy, \varphi\rangle$ is $\sigma(A^{**}, A^*)$-continuous. By Jordan decomposition, the mapping $A^{**} \ni x \mapsto xy \in A^{**}$ is $\sigma(A^{**}, A^*)$-continuous. Hence, for any $x, y \in A^{**}$, we have $(xy)^* = y^*x^*$. Consequently, A^{**} is a Banach *-algebra.

Let $(e_\iota)_\iota$ be an approximate identity of A. Let 1 be a cluster point of $(e_\iota)_\iota$ with respect to the $\sigma(A^{**}, A^*)$-topology; then, for any $x \in A$, we have $1x = x1 = x$. Therefore we have $1x = x1 = x$ for all $x \in A^{**}$, that is, 1 is an identity of A^{**}. □

Let A be a C^*-algebra and φ a state of A. φ is self-adjoint in A^{**}. It holds that, for any elements x and y of the unit ball of A^{**},

$$\begin{aligned}|\varphi(y^*y) - \varphi(x^*x)| &\leq |\varphi(y^*(y-x))| + |\varphi((y-x)^*x)| \\ &= |\varphi(y^*(y-x))| + |\varphi(x^*(y-x))| \\ &\leq 2 \sup_{\|a\|\leq 1} |(\varphi a)(y-x)|.\end{aligned}$$

Since the set $\{\varphi a \mid a \in A^{**}, \|a\| \leq 1\}$ is a $\sigma(A^*, A^{**})$-compact balanced convex set, the function $x \mapsto \varphi(x^*x)$ is continuous on the unit ball of A^{**} with respect to the $\tau(\mathscr{M}^{**}, \mathscr{M}^*)$-topology. Hence φ is positive on A^{**}. We can define the seminorms p_φ and p^*_φ on A^{**}:

$$p_\varphi(x) = \varphi(x^*x)^{1/2} \qquad \text{and} \qquad p^*_\varphi(x) = \varphi(xx^*)^{1/2}.$$

We call the topology defined by all p_φ (resp., all p_φ and p^*_φ) the σ-strong topology (resp., the σ-strong* topology). By Jordan decomposition, the σ-strong topology is finer than the $\sigma(A^{**}, A^*)$-topology. p_φ and p^*_φ are continuous on the unit ball with respect to the $\tau(A^{**}, A^*)$-topology. Hence, for a σ-strongly* continuous linear form ψ on A^{**}, the intersection of $\ker\psi$ and the unit ball is $\tau(A^{**}, A^*)$-closed and so $\sigma(A^{**}, A^*)$-closed. By the Banach theorem, $\ker\psi$ is $\sigma(A^{**}, A^*)$-closed, and hence ψ is $\sigma(A^{**}, A^*)$-continuous, that is, $\psi \in A^*$. Therefore the σ-strong topology and σ-strong* topology are compatible with the duality $\langle A^{**}, A^*\rangle$. Hence, the unit ball of A^{**} is σ-strongly closed and the unit ball of A is σ-strongly* dense in the unit ball of A^{**}.

Notice that A^{**} has an identity.

Lemma 3. *Let A be a C^*-algebra and $S(A)$ the state space of A. Then it holds that, for any self-adjoint element x of A^{**},*

$$\|x\| = \sup_{\varphi\in S(A)} |\varphi(x)|.$$

Proof. Let x be a self-adjoint element of A^{**} and δ an arbitrary positive real number; then there exists an element φ of A^* such that $\varphi(x) \geq \|x\| - \delta$ and $\|\varphi\| \leq 1$. We have $\varphi(x) = 2^{-1}(\varphi+\varphi^*)(x)$. Put $\psi = 2^{-1}(\varphi+\varphi^*)$ and let $\psi = \psi_+ - \psi_-$ be a Jordan decomposition of ψ; then we have

$$\begin{aligned}\varphi(x) = \psi(x) &\leq |\psi_+(x)| + |\psi_-(x)| \\ &\leq (\|\psi_+\| + \|\psi_-\|) \sup_{\varphi\in S(A)} |\varphi(x)| = \|\psi\| \sup_{\varphi\in S(A)} |\varphi(x)| \\ &\leq \sup_{\varphi\in S(A)} |\varphi(x)| \leq \|x\|.\end{aligned}$$

Therefore we obtain $\|x\| = \sup_{\varphi\in S(A)} |\varphi(x)|$. □

Lemma 4. *Let A be a C^*-algebra. Then, for any self-adjoint element x of A^{**}, we have $\|x^2\| = \|x\|^2$ and so $\|x\| = r(x)$, where $r(x)$ denotes the spectral radius of x.*

Proof. By Lemma 3 and the Cauchy-Schwarz inequality, for any self-adjoint element $x \in A^{**}$, we have

$$\|x\| \leq \sup_{\varphi \in S(A)} \varphi(x^2)^{1/2} \leq \|x^2\|^{1/2},$$

and so $\|x^2\| = \|x\|^2$. Therefore we have

$$r(x) = \lim_{n \to \infty} \|x^{2^n}\|^{2^{-n}} = \|x\|.$$

□

Let B be a commutative Banach *-subalgebra of A^{**} and the mapping $B \ni x \mapsto \hat{x} \in C_0(\Omega)$ the Gelfand representation of B. If x and y are self-adjoint elements of B and $\hat{x} = \hat{y}$, then we have $x = y$. For, since the spectral radius in B coincides with the spectral radius in A^{**}, we have

$$\|x - y\| = r(x - y) = \sup_{\omega \in \Omega} |\hat{x}(\omega) - \hat{y}(\omega)| = 0$$

and so $x = y$.

Lemma 5. *Let A be a Banach algebra with an identity and assume that A is the dual space of a normed space E and the multiplication in A is separately continuous with respect to the $\sigma(A, E)$-topology. Let B and λ be a subset of A and a complex number, respectively, such that $\sup_{y \in B} \|(\lambda 1 - y)^{-1}\| < +\infty$. If x belongs to the $\tau(A, E)$-closure of B, then we have $\lambda \notin \mathrm{Sp}(x)$ and $(\lambda 1 - x)^{-1} = \lim_{y \to x, y \in B} (\lambda 1 - y)^{-1}$ with respect to the $\sigma(A, E)$-topology.*

Proof. Let $\mathfrak{F}$ be an ultrafilter on B which converges to x with respect to the $\tau(A, E)$-topology. Since the image of an ultrafilter under a function is an ultrafilter base, there is a limit $a = \lim_{y, \mathfrak{F}} (\lambda 1 - y)^{-1}$ with respect to the $\sigma(A, E)$-topology. For any $\varphi \in E$ and $z \in A$, φz is in E and the mapping $A \ni z \mapsto \varphi z \in E$ is continuous with respect to the $\sigma(A, E)$-topology and $\sigma(E, A)$-topology, so that $\{\varphi z \mid \|z\| \leq 1\}$ is compact. Hence we have $\lim_{y, \mathfrak{F}} (\lambda 1 - y)^{-1}((\lambda 1 - y) - (\lambda 1 - x)) = 0$ with respect to the $\sigma(A, E)$-topology. Therefore we obtain $1 - a(\lambda 1 - x) = 0$ and so $a = (\lambda 1 - x)^{-1}$. Hence we have $(\lambda 1 - x)^{-1} = \lim_{y \to x, y \in B} (\lambda 1 - y)^{-1}$. □

Lemma 6. *Let A be a C^*-algebra. Then we have $\mathrm{Sp}(x^* x) \subset \mathbf{R}_+$ for every element $x \in A^{**}$.*

Proof. Since the function $x \mapsto x^* x$ is σ-strongly* continuous on the unit ball of A^{**}, for any x in the unit ball of A^{**}, $x^* x$ belongs to σ-strong* closure of the positive portion of A and so belongs to the closure of the positive portion of A with respect to the $\tau(A^{**}, A^*)$-topology. For any $\lambda \notin \mathbf{R}_+$ and positive element $y \in A$, we have $\|(\lambda 1 - y)^{-1}\| \leq d(\lambda, \mathbf{R}_+)^{-1} < +\infty$. Hence, by Lemma 5, we have $\lambda \notin \mathrm{Sp}(x^* x)$ and so $\mathrm{Sp}(x^* x) \subset \mathbf{R}_+$. □

If $\mathrm{Sp}(x) \subset \mathbf{R}$ and B is a Banach subalgebra of A^{**} containing x and 1, then we have $\mathrm{Sp}_B(x) = \mathrm{Sp}(x)$. For, if $\mathrm{Sp}_B(x) \not\subset \mathbf{R}$, then there is a number $\lambda \in \mathrm{Sp}_B(x)$ such that $d(\lambda, \mathrm{R}) = \sup_{\mu \in \mathrm{Sp}_B(x)} d(\mu, \mathbf{R}) > 0$. λ is a boundary point of $\mathrm{Sp}_B(x)$ and so is a boundary point of $\mathrm{Sp}(x)$. Hence we have $\lambda \in \mathrm{Sp}(x) \subset \mathbf{R}$, which is a contradiction. Therefore $\mathrm{Sp}_B(x)$ is included in $\mathbf{R}$ and so coincides with $\mathrm{Sp}(x)$.

Let $\mathscr{S}$ and A_+^{**} denote the unit ball of the second dual A^{**} of a C^*-algebra A and the set of all x^*x with $x \in A^{**}$, respectively.

Lemma 7. *Let A be a C^*-algebra and x a self-adjoint element of A^{**} with $\mathrm{Sp}(x) \subset \mathbf{R}_+$. Then there exists a unique self-adjoint element y of A^{**}, denoted by $x^{1/2}$, such that $x = y^2$ and $\mathrm{Sp}(y) \subset \mathbf{R}_+$. Therefore A_+^{**} coincides with the set of all self-adjoint elements x such that $\mathrm{Sp}(x) \subset \mathbf{R}_+$. Furthermore, the function $A_+^{**} \cap \mathscr{S} \ni x \mapsto x^{1/2} \in A_+^{**} \cap \mathscr{S}$ is σ-strongly continuous.*

Proof. Let x be a self-adjoint element in A^{**} with $\mathrm{Sp}(x) \subset \mathbf{R}_+$ and $\|x\| \leq 1$. There exists a sequence $(p_n)_n$ of polynomials with real coefficients such that

$$\lim_{n\to\infty} \sup_{0 \leq t \leq 1} |p_n(t) - t^{1/2}| = 0.$$

Let B be the commutative Banach *-subalgebra of A^{**} generated by x and 1, and the mapping $B \ni y \mapsto \hat{y} \in C(\Omega)$ the Gelfand representation. By Lemma 4, it holds that

$$\|p_n(x) - p_m(x)\| = \sup_{\omega \in \Omega} |p_n(\hat{x}(\omega)) - p_m(\hat{x}(\omega))| \leq \sup_{0 \leq t \leq 1} |p_n(t) - p_m(t)|.$$

Hence the sequence $(p_n(x))_n$ is a Cauchy sequence and so converges in norm to some self-adjoint element y. Since $\hat{y}(\omega) = \lim_{n\to\infty} p_n(\hat{x}(\omega)) = \hat{x}(\omega)^{1/2}$, we have $\widehat{y^2} = \hat{y}^2 = \hat{x}$ and so $y^2 = x$ and $\mathrm{Sp}(y) = \hat{y}(\Omega) \subset \mathbf{R}_+$. If z is a self-adjoint element, $x = z^2$ and $\mathrm{Sp}(z) \subset \mathbf{R}_+$, then z commutes with x. Hence there is a commutative Banach *-subalgebra C containing x, z and 1. Since $y \in C$, considering the Gelfand representation of C, we have $\hat{y} = \hat{x}^{1/2} = \hat{z}$ and so $y = z$.

$A_+^{**} \cap \mathscr{S} \ni x \mapsto p_n(x) \in A^{**}$ is σ-strongly continuous and

$$\sup_{x \in A_+^{**} \cap \mathscr{S}} \|p_n(x) - x^{1/2}\| = \sup_{x \in A_+^{**} \cap \mathscr{S}} \sup_{\omega \in \Omega} |p_n(\hat{x}(\omega)) - \hat{x}(\omega)^{1/2}| \leq \sup_{0 \leq t \leq 1} |p_n(t) - t^{1/2}|.$$

Since the limit of a sequence of continuous functions with respect to the topology of uniform convergence is continuous, the function $A_+^{**} \cap \mathscr{S} \ni x \mapsto x^{1/2} \in A_+^{**} \cap \mathscr{S}$ is σ-strongly continuous. □

Remark. Let A be a C^*-algebra and (e_ι) an approximate identity of A. Since the norm of A^{**} is lower semi-continuous with respect to the $\sigma(A^{**}, A^*)$-topology, we have, for any

$x \in A$ and $\lambda \in \mathbf{C}$,

$$\begin{aligned}\|x+\lambda 1\|^2 &\leq \liminf_\iota \|x+\lambda e_\iota\|^2\\ &\leq \liminf_\iota (\|x - xe_\iota\| + \|xe_\iota + \lambda e_\iota\|)^2\\ &= \liminf_\iota \|xe_\iota + \lambda e_\iota\|^2 = \liminf_\iota \|(xe_\iota + \lambda e_\iota)^*(xe_\iota + \lambda e_\iota)\|\\ &= \liminf_\iota \|e_\iota (x+\lambda 1)^*(x+\lambda 1)e_\iota\| \leq \|(x+\lambda 1)^*(x+\lambda 1)\|.\end{aligned}$$

The subalgebra $A + \mathbf{C}1$ of A^{**} is therefore a C^*-algebra.

Theorem 8. *The second dual of a C^*-algebra is a C^*-algebra.*

Proof. Let A be a C^*-algebra. For any $x \in A^{**}_+$, we have $\mathrm{Sp}(1+x) \subset [1, +\infty)$ and so $\mathrm{Sp}((1+x)^{-1}) \subset (0,1]$. Hence, by Lemma 4, we have $\|(1+x)^{-1}\| = r((1+x)^{-1}) \leq 1$. Therefore the function $A^{**}_+ \ni x \mapsto (1+x)^{-1} \in A^{**}$ is σ-strongly continuous. Since the function $\mathscr{S} \ni x \mapsto x^*x \in A^{**}_+ \cap \mathscr{S}$ is σ-strongly* continuous, for a positive natural number n, the function $\mathscr{S} \ni x \mapsto (1 + n(x^*x)^{1/2})^{-1} \in A^{**}$ is σ-strongly* continuous, in virtue of Lemma 7. Therefore the function $\mathscr{S} \ni x \mapsto x(n^{-1}1 + (x^*x)^{1/2})^{-1} \in A^{**}$ is continuous with respect to the σ-strong* topology and σ-strong topology. By considering spectrum or the above remark, for any $x \in A$, we have $\|x(n^{-1}1 + (x^*x)^{1/2})^{-1}\| \leq 1$. Hence we obtain $\|x(n^{-1}1 + (x^*x)^{1/2})^{-1}\| \leq 1$ for every $x \in \mathscr{S}$. Since

$$x - x(n^{-1}1 + (x^*x)^{1/2})^{-1}(x^*x)^{1/2} = n^{-1}x(n^{-1}1 + (x^*x)^{1/2})^{-1},$$

we have

$$\|x - x(n^{-1}1 + (x^*x)^{1/2})^{-1}(x^*x)^{1/2}\| \leq n^{-1}.$$

Therefore it follows that

$$\begin{aligned}\|x\| &= \lim_{n\to\infty} \|x(n^{-1}1 + (x^*x)^{1/2})^{-1}(x^*x)^{1/2}\|\\ &\leq \|(x^*x)^{1/2}\| = \|x^*x\|^{1/2},\end{aligned}$$

so that $\|x^*x\| = \|x\|^2$. Consequently, A^{**} is a C^*-algebra. □

2. Identities in W^*-algebras

In [1], we used the projection of the second dual of a W^*-algebra of norm one, however, in the following, we do not need the second dual of a W^*-algebra.

Theorem 9. *Any W^*-algebra has an identity.*

Proof. Let $\mathscr{M}$ be a W^*-algebra and $(e_\iota)_\iota$ be its approximate identity. There exists a cluster point 1 in $\mathscr{M}$ of $(e_\iota)_\iota$ with respect to the σ-weak topology. For $\kappa \geq \iota$, we have $-e_\kappa \leq 2e_\iota - e_\kappa \leq e_\iota$ and so $\|2e_\iota - e_\kappa\| \leq 1$. Hence we have $\|2e_\iota - 1\| \leq 1$, so that $|\varphi(2e_\iota - 1)| \leq 1$ for every state φ of $\mathscr{M}$. Since $\lim_\iota \varphi(e_\iota) = 1$, we have $|2 - \varphi(1)| \leq 1$. Since $|\varphi(1)| \leq 1$, we obtain $\varphi(1) = 1 = \lim_\iota \varphi(e_\iota)$. Since the state space of $\mathscr{M}$ is algebraically total in the dual space of $\mathscr{M}$, 1 is a limit of $(e_\iota)_\iota$ with respect to the $\sigma(\mathscr{M}, \mathscr{M}^*)$-topology. Therefore, for any $x \in \mathscr{M}$ and $\varphi \in \mathscr{M}^*$, it holds that

$$\langle 1x, \varphi\rangle = \langle 1, x\varphi\rangle = \lim_\iota \langle e_\iota, x\varphi\rangle = \lim_\iota \langle e_\iota x, \varphi\rangle = \langle x, \varphi\rangle,$$

so that $1x = x$. Similarly, we have $x1 = x$, and hence, 1 is an identity of $\mathscr{M}$. □

Lemma 10. *Let $\mathscr{M}$ be a W^*-algebra and ε the projection of the second dual $\mathscr{M}^{**}$ onto $\mathscr{M}$ of norm one. Then ε is positive and so self-adjoint.*

Proof. $\mathscr{M}^{**}$ is a C^*-algebra. Since, for any state φ of $\mathscr{M}$, $\varphi \circ \varepsilon(1) = 1$, $\varphi \circ \varepsilon$ is positive and so ε is positive. Therefore ε is trivially self-adjoint. □

Theorem 11. *The involution in a W^*-algebra $\mathscr{M}$ is σ-weakly continuous.*

Proof. Let ε be the canonical projection of $\mathscr{M}^{**}$ onto $\mathscr{M}$ of norm one; then, by Lemma 10, there is a commutative diagram as follows:

$$\begin{array}{ccc} \mathscr{M}^{**} & \xrightarrow{\text{involution}} & \mathscr{M}^{**} \\ {\scriptstyle\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle\varepsilon} \\ \mathscr{M} & \xrightarrow[\text{involution}]{} & \mathscr{M} \end{array}$$

Since the involution in $\mathscr{M}^{**}$ is $\sigma(\mathscr{M}^{**}, \mathscr{M}^*)$-continuous, the involution in $\mathscr{M}$ is σ-weakly continuous. □

3. The Kaplansky density theorem

By Theorem 8, we immediately see the Kaplansky density theorem. Also, we can show the Kaplansky density theorem in virtue of polar decomposition.

Proposition 12. *Let $\mathscr{M}$ and $\mathscr{N}$ be two W^*-algebras and Φ a σ-weakly continuous *-homomorphism of $\mathscr{M}$ into $\mathscr{N}$. Then $\Phi(\mathscr{M})$ is σ-weakly closed and the unit ball of $\Phi(\mathscr{M})$ coincides with the image of the unit ball of $\mathscr{M}$ under Φ.*

Proof. Let j be the canonical mapping of $\mathscr{M}$ onto $\mathscr{M}/\ker\Phi$; then there exists a *-isomorphism Ψ of $\mathscr{M}/\ker\Phi$ into $\mathscr{N}$ such that $\Phi = \Psi \circ j$. Since $\mathscr{M}/\ker\Phi$ is a C^*-algebra, Ψ is an isometry. The image of the open unit ball of $\mathscr{M}$ under j coincides with the open unit ball of $\mathscr{M}/\ker\Phi$. Hence the image of the open unit ball of $\mathscr{M}$ under Φ coincides with the open unit ball of $\Phi(\mathscr{M})$. Since the closed unit ball of $\mathscr{M}$ is σ-weakly compact, the image of the closed unit ball of $\mathscr{M}$ under Φ is σ-weakly compact and so coincides with the closed unit ball of $\Phi(\mathscr{M})$. Therefore $\Phi(\mathscr{M})$ is σ-weakly closed. □

Let $\mathscr{M}$ be a W^*-algebra and V a uniformly dense linear subspace of $\mathscr{M}_*$ such that $\varphi^*, a\varphi$ and φa belong to V for every $\varphi \in V$ and $a \in \mathscr{M}$.

Lemma 13. *Let $\mathscr{M}$ and V be as above and A a $\sigma(\mathscr{M}, V)$-dense *-subalgebra of $\mathscr{M}$. Then A is σ-weakly dense in $\mathscr{M}$.*

Proof. The self-adjoint portion A^s of A is dense in $\mathscr{M}^s$ with respect to the $\tau(\mathscr{M}, V)$-topology. For any complex number $\lambda \notin \mathbf{R}$ and self-adjoint element y, we have

$$\|(\lambda 1 - y)^{-1}\| \leq d(\lambda, \mathbf{R})^{-1} < +\infty,$$

where $d(\lambda, \mathbf{R})$ denotes the distance between λ and $\mathbf{R}$. By Lemma 5, we have $(\lambda 1 - x)^{-1} = \lim_{y \to x, y \in A^s} (\lambda 1 - y)^{-1}$ for every $x \in \mathscr{M}^s$. Hence, $(\lambda 1 - x)^{-1}$ belongs to the σ-weak closure of $A + \mathbf{C}1$ and also does $\lambda 1 - x$. Since $x = (\lambda 1 - x)(\lambda(\lambda 1 - x)^{-1} - 1)$ and $\lambda(\lambda 1 - x)^{-1} - 1 = \lim_{y \to x, y \in A^s} (\lambda 1 - y)^{-1} y$, x belongs to the σ-weak closure of A. Therefore A is σ-weakly dense in $\mathscr{M}$. □

Theorem 14 (Kaplansky). *Let $\mathscr{M}$ and V be as above and A a *-subalgebra of $\mathscr{M}$ which is $\sigma(\mathscr{M}, V)$-dense in $\mathscr{M}$. Then the unit ball of A is $\tau(\mathscr{M}, \mathscr{M}_*)$-dense in the unit ball of $\mathscr{M}$.*

Proof. We may assume, without loss of generality, that A is a C^*-algebra. Let id denote the identity mapping of A into $\mathscr{M}$ and Φ the transpose mapping of the restriction ${}^t\mathrm{id}|\mathscr{M}_*$; then Φ is a continuous *-homomorphism of A^{**} equipped with the $\sigma(A^{**}, A^*)$-topology into $\mathscr{M}$ equipped with the σ-weak topology. We regard Φ as an extension of id. By Theorem 8, A^{**} is a W^*-algebra. By Proposition 12, $\Phi(A^{**})$ is σ-weakly closed and the image of the unit ball of A^{**} under Φ coincides with the unit ball of $\Phi(A^{**})$. Since $\Phi(A^{**})$ is $\sigma(\mathscr{M}, V)$-dense in $\mathscr{M}$, $\Phi(A^{**})$ coincides with $\mathscr{M}$, in virtue of Lemma 13. Since the unit ball of A is $\sigma(A^{**}, A^*)$-dense in the unit ball of A^{**}, the unit ball of A is σ-weakly dense in the unit ball of $\mathscr{M}$ and so $\tau(\mathscr{M}, \mathscr{M}_*)$-dense in the unit ball of $\mathscr{M}$. □

Proposition 15 (Polar Decomposition). *Let $\mathscr{M}$ be a W^*-algebra. For any element x of $\mathscr{M}$, there exists one and only one partial isometry v in $\mathscr{M}$ such that $x = v|x|$ and $v^*v = s(|x|)$.*

Proof. Put $v_n = x(n^{-1}1 + |x|)^{-1}$ for each positive natural number n; then we have $\|v_n\| \leq 1$. Since $|v_n| = (n^{-1}1 + |x|)^{-1}|x|$, $(|v_n|)_n$ is increasing and so σ-strongly convergent. Since $s(|x|)|v_n| = |v_n|$, we have $s(|x|)\lim_{n\to\infty}|v_n| = \lim_{n\to\infty}|v_n|$. Since $|x| - |x||v_n| = n^{-1}|v_n|$, we have $|x| = |x|\lim_{n\to\infty}|v_n|$. Hence we have $s(|x|)(1 - \lim_{n\to\infty}|v_n|) = 0$ and so $\lim_{n\to\infty}|v_n| = s(|x|)$. Since $v_n^*v_m = |v_n||v_m|$, we have $(v_n - v_m)^*(v_n - v_m) = (|v_n| - |v_m|)^2$. Hence $(v_n)_n$ is a Cauchy sequence with respect to the σ-strong topology. Since the unit ball of $\mathscr{M}$ is complete with respect to the σ-strong topology, $(v_n)_n$ converges σ-strongly to some element $v \in \mathscr{M}$. Since $x - v_n|x| = n^{-1}v_n$, we obtain $x = v|x|$. Since

$$x^*v = \lim_{n\to\infty} x^*v_n = \lim_{n\to\infty} |x||v_n| = |x|s(|x|) = |x|,$$

we have $|x|v^*v = |x|$ and so $v^*v = s(|x|)$. Therefore v is a partial isometry.

Let w be a partial isometry in $\mathscr{M}$ such that $x = w|x|$ and $w^*w = s(|x|)$; then we have

$$v = \lim_{n\to\infty} x(n^{-1}1 + |x|)^{-1} = \lim_{n\to\infty} w|x|(n^{-1}1 + |x|)^{-1} = ws(|x|) = w.$$

$\square$

Another proof of the Kaplansky density theorem. Let $\mathfrak{F}$ be a filter on A converging to a partial isometry $v \in \mathscr{M}$ with respect to the $\tau(\mathscr{M}, V)$-topology. It holds that, for any $x \in \mathscr{M}$,

$$\begin{aligned}(1 + xx^*)^{-1} - (1 + vv^*)^{-1} &= (1 + xx^*)^{-1}(vv^* - xx^*)(1 + vv^*)^{-1} \\ &= ((1 + xx^*)^{-1}x(x - v)^* + (1 + xx^*)^{-1}(x - v)v^*)(1 + vv^*)^{-1}.\end{aligned}$$

Since $\|2(1 + xx^*)^{-1}x\| \leq 1$, we have $(1 + vv^*)^{-1} = \lim_{x,\mathfrak{F}}(1 + xx^*)^{-1}$. Since $\lim_{x,\mathfrak{F}}(1 + xx^*)^{-1}(x - v) = 0$, we obtain $v = 2(1 + vv^*)^{-1}v = \lim_{x,\mathfrak{F}} 2(1 + xx^*)^{-1}x$. Hence v belongs to the σ-weak closure $\overline{A \cap \mathscr{S}}$ of $A \cap \mathscr{S}$. In particular, any projection in $\mathscr{M}$ belongs to $\overline{A \cap \mathscr{S}}$. Hence, by spectral decomposition, any positive element in $\mathscr{S}$ belongs to $\overline{A \cap \mathscr{S}}$. Therefore, by polar decomposition, we obtain $\mathscr{S} \subset \overline{A \cap \mathscr{S}}$.

References

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